

ON TENSOR SPACES FOR ROOK MONOID ALGEBRAS

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ABSTRACT. Let $m, n \in \mathbb{N}$, and V be a m -dimensional vector space over a field F of characteristic 0. Let $U = F \oplus V$ and R_n be the rook monoid. In this paper, we construct a certain quasi-idempotent in the annihilator of $U^{\otimes n}$ in FR_n , which comes from some one-dimensional two-sided ideal of rook monoid algebra. We show that the two-sided ideal generated by this element is indeed the whole annihilator of $U^{\otimes n}$ in FR_n .

1. INTRODUCTION

Let $m, n \in \mathbb{N}$ and V be a m -dimensional vector space over a field F of characteristic 0. Let $U = F \oplus V$ and $\mathrm{GL}(V)$ be the general linear group over V . We further consider F as the trivial $\mathrm{GL}(V)$ -module. This allows us to consider U and hence $U^{\otimes n}$ as a $\mathrm{GL}(V)$ -module. Let R_n be the rook monoid, see Section 2 for precise definition. By [13], there is a left action of R_n on $U^{\otimes n}$ which commutes with the left action of $\mathrm{GL}(V)$. Let φ, ψ be the natural algebra homomorphisms:

$$\begin{aligned}\varphi &: FR_n \rightarrow \mathrm{End}_{\mathrm{GL}(V)}(U^{\otimes n}), \\ \psi &: F\mathrm{GL}(V) \rightarrow \mathrm{End}_{FR_n}(U^{\otimes n}),\end{aligned}$$

respectively. The following results are proved by Solomon [13, Theorem 5.10 and Corollary 5.18].

Theorem 1.1. 1) Both φ and ψ are surjective;
2) if $m \geq n$, then φ is an isomorphism.

The above theorem is an analogue, for R_n and $\mathrm{GL}(V)$, of the Schur-Weyl duality for symmetric group \mathfrak{S}_n and general linear group $\mathrm{GL}(V)$. When $m < n$, the algebra homomorphism φ is in general not injective. Therefore it is natural to ask how to describe the kernel of the homomorphism φ , i.e., the annihilator of $U^{\otimes n}$ in the algebra FR_n . This question is closely related to the invariant theory, see [2, 15].

Let G be an algebraic subgroup of $\mathrm{GL}(V)$ and M be a G -module. One formulation of the invariant theory for G is to describe the endomorphism algebra $\mathrm{End}_G(M^{\otimes n})$. It should be remarked that, in the classical invariant theory, G -module M is usually set as the natural representation V . The first fundamental theorem of invariant theory provides generators of $\mathrm{End}_G(M^{\otimes n})$ and the second fundamental theorem of invariant theory describes all the relations among the generators. From this point of view, the above Theorem 1.1 can be seen as the first fundamental theorem of invariant theory for general linear group $\mathrm{GL}(V)$ about the module U . Therefore, it is desirable to give out the second fundamental theorem,

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i.e., a characterization of the annihilator ideal of $U^{\otimes n}$ in rook monoid algebra FR_n by its standard generators.

The purpose of this article is to answer the above question. Recently Hu and the author [6] proved the second fundamental theorem for symplectic group and Lehrer-Zhang in [8] gave out the second fundamental theorem for orthogonal group, where they deeply used the different versions of invariant theory. In both symplectic and orthogonal cases, the annihilator of n -tensor space in the specialized Brauer algebra is generated by an explicitly described quasi-idempotent. Motivated by the articles [6, 8], we construct a certain quasi-idempotent Y_{m+1} (see Section 4) in $\text{Ker } \varphi$ and prove that

Theorem 1.2. *With notations as above, if $m < n$, we have*

$$\text{Ann}_{FR_n}(U^{\otimes n}) = \langle Y_{m+1} \rangle.$$

We would like to point out that an analogue of Theorem 1.1 for orthogonal group and rook Brauer algebra (also called partial Brauer algebra) was obtained by Halverson-delMas [5] and Martin-Mazorchuk [9] independently. We conjecture that there exists an analogue of Theorem 1.2 for rook Brauer algebra and will consider this question in a future separate article.

The content of this article is organized as follows. In Section 2 we recall some basic knowledge about the structure and representation theory of rook monoids as well as some combinatorics which are needed later. In Section 3 we construct the one-dimensional two-sided ideals in the rook monoid algebra FR_n . Furthermore, we can get the block decomposition of FR_n by the theory of Specht modules. In Section 4 we prove our main result Theorem 1.2 and the proof will be proceed in three steps.

2. PRELIMINARIES

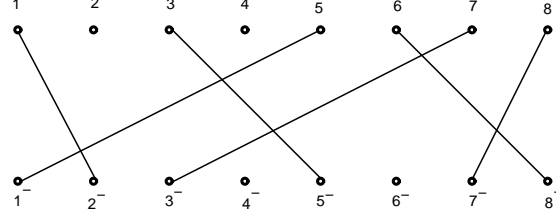
2.1. Rook monoid. Let R_n be the set of all $n \times n$ matrices that contain at most one entry equal to 1 in each row and column and zeros elsewhere. With the operation of matrix multiplication, R_n has the structure of a monoid. The monoid R_n is known both as the *rook monoid* and the *symmetric inverse semigroup* [12]. The number of rank r matrices in R_n is $\binom{n}{r}^2 r!$ and hence the rook monoid has a total of $\sum_{r=0}^n \binom{n}{r}^2 r!$ elements.

A presentation of the rook monoid R_n is given in [7] which is more helpful for us (see also [5, Section 2]). The rook monoid R_n is generated by $s_1, \dots, s_{n-1}, p_1, \dots, p_n$ subject to the following relations:

$$\begin{aligned} s_i^2 &= 1, & 1 \leq i \leq n-1, \\ s_i s_j &= s_j s_i, & |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & 1 \leq i \leq n-2, \\ p_i^2 &= p_i, & 1 \leq i \leq n, \\ p_i p_j &= p_j p_i, & i \neq j, \\ s_i p_i &= p_{i+1} s_i, & 1 \leq i \leq n-1, \\ s_i p_j &= p_j s_i, & |i-j| > 1, \\ p_i s_i p_i &= p_i p_{i+1}, & 1 \leq i \leq n-1. \end{aligned}$$

It is clear that the symmetric group $\mathfrak{S}_n \subseteq R_n$.

Now we recall another presentation of R_n by rook n -diagram (see [5, 7]). A rook n -diagram is a graph consisting of two rows each with n vertices such that each vertex in the top row is connected to at most one vertex in the bottom row. We denote Rd_n the set of all rook n -diagram. For each rook n -diagram D , we shall label the vertices in the top row of D by $1, 2, \dots, n$ from left to right, and label the vertices in the bottom row of D by $1^-, 2^-, \dots, n^-$ also from left to right. For example, let D be the following rook diagram



Then $D \in \text{Rd}_8$. For a rook n -diagram, the vertices which are not incident to an edge are called *isolated vertices*. The multiplication of two rook n -diagram is defined using natural concatenation of diagrams. Precisely, we compose two rook n -diagrams D_1, D_2 by identifying the bottom row of vertices in D_1 with the top row of vertices in D_2 . The result is also a rook n -diagram and defined as the multiplication $D_1 \cdot D_2$.

There is a connection between the above two presentations of rook monoids. For each integer $1 \leq i < n$, the standard generator s_i corresponds to the rook n -diagram with edges connecting the vertices i (resp. $i+1$) on the top row with $(i+1)^-$ (resp. i^-) on the bottom row, and all other edges being vertical, connecting the vertices k and k^- for all $k \neq i, i+1$. For each integer $1 \leq j \leq n$, the standard generator p_j corresponds to the rook n -diagram with isolated vertices j and j^- , and vertical edges $\{k, k^-\}$ for all $k \neq j$. For an integer r with $0 \leq r \leq n$, we define

$$\mathcal{D}_r := \{d \in \mathfrak{S}_n \mid (1)d < (2)d < \dots < (r)d, (r+1)d < \dots < (n)d\}.$$

Note that $\mathcal{D}_0 = \{1\}$ and \mathcal{D}_r is the set of distinguished right coset representatives of $\mathfrak{S}_{(r, n-r)}$ in \mathfrak{S}_n . It is helpful to point out that we consider the elements of symmetric group as *right* permutations. From this point of view, the composition of permutations coincides with the multiplication of rook diagrams. Let F be a field of characteristic 0. The following proposition follows directly from the diagrammatic multiplication of rook monoid algebras.

Proposition 2.1. *For each $D \in \text{Rd}_n$ there exists a unique quadruple (d_1, d_2, r, σ) with $0 \leq r \leq n$, $d_1, d_2 \in \mathcal{D}_r$, $\sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$ and such that $D = d_1^{-1} p_1 p_2 \dots p_r \sigma d_2$. In particular, the elements in the following set*

$$\{d_1^{-1} p_1 p_2 \dots p_r \sigma d_2 \mid 0 \leq r \leq n, d_1, d_2 \in \mathcal{D}_r, \sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}\}$$

form a basis of the rook monoid algebra FR_n and it coincides with the natural basis given by rook n -diagram.

Note that in the above proposition, the element $d_1^{-1} p_1 p_2 \dots p_r \sigma d_2$ corresponds to the rook n -diagram with the isolated vertices $(1)d_1, (2)d_1, \dots, (r)d_1$ in the top row, the isolated vertices $((1)d_2)^-, ((2)d_2)^-, \dots, ((r)d_2)^-$ in the bottom row, and edges connecting $(j)d_1$ with $((j)\sigma d_2)^-$ for $j = r+1, r+2, \dots, n$.

As is predicated in the introduction, there is a left action of FR_n on the n -tensor space $U^{\otimes n}$ which commutes with the left action of $\text{GL}(V)$. We now recall

the definition of this action. Let $\delta_{i,j}$ denote the value of the usual Kronecker delta. We fix a basis $\{v_1, v_2, \dots, v_m\}$ of V and a basis $\{v_0\}$ of F such that

$$U^{\otimes n} = F - \text{Span}\{v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n} \mid i_j = 0, 1, \dots, m\}.$$

The left action of FR_n on $U^{\otimes n}$ is defined on generators by (see [13])

$$\begin{aligned} s_j(v_{i_1} \otimes \dots \otimes v_{i_n}) &:= v_{i_1} \otimes \dots \otimes v_{i_{j-1}} \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_{j+2}} \otimes \dots \otimes v_{i_n}, \\ p_j(v_{i_1} \otimes \dots \otimes v_{i_n}) &:= \delta_{i_j,0} v_{i_1} \otimes \dots \otimes v_{i_{j-1}} \otimes v_0 \otimes v_{i_{j+1}} \otimes \dots \otimes v_{i_n}. \end{aligned}$$

2.2. Specht module. Munn demonstrated the representations of rook monoids stemmed from his work on the general theory of representations of finite semigroups [10, 11]. He computed the irreducible characters of R_n using irreducible characters of symmetric group \mathfrak{S}_r ($1 \leq r \leq n$) and showed that the rook monoid algebra FR_n is semisimple. Motivated by Munn's work and the theory of Specht modules of symmetric groups, Grood [4] studied the theory of Specht modules of FR_n which we recall here. It should be pointed out that although Grood worked on the complex field \mathbb{C} , it is clear that all the results in [4] also hold for an arbitrary field F of characteristic 0.

A partition of r is a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_{i \geq 1} \lambda_i = r$. In this case, we write $\lambda \vdash r$ and $|\lambda| = r$. The length of λ , denoted $\ell(\lambda)$, is the maximum subscript j such that $\lambda_j > 0$. The Young diagram of λ is defined to be the set

$$[\lambda] := \{(i, j) \mid 1 \leq j \leq \lambda_i\}.$$

A λ -tableau is a bijection $\mathbf{t} : [\lambda] \rightarrow \{1, 2, \dots, r\}$.

Definition 2.2. Let $\lambda \vdash r$ with $0 \leq r \leq n$. An n -tableau of shape λ , also called a λ_r^n -tableau, is a bijection $\mathbf{t} : [\lambda] \rightarrow S$, where S is a subset of r distinct elements of the set $\{1, 2, \dots, n\}$.

Let \mathbf{t} be a λ_r^n -tableau. We write $\mathbf{t}_{ij} = \mathbf{t}(i, j)$, the entry contained in the box of the i -th row and j -th column of \mathbf{t} by Grood's notations. The *content* of \mathbf{t} , denoted $\text{cont}(\mathbf{t})$, is the image of \mathbf{t} . For a given λ_r^n -tableau \mathbf{t} , let \mathcal{R}_i (resp. \mathcal{C}_j) be the set of entries in the i -th row (resp. the j -th column) of \mathbf{t} . Two λ_r^n -tableaux \mathbf{t} and \mathbf{s} are called *row-equivalent* if the corresponding rows of the two tableaux contain the same entries. In other words, the set \mathcal{R}_i of \mathbf{t} coincides with that of \mathbf{s} for $1 \leq i \leq \ell(\lambda)$. In this case, we write $\mathbf{t} \sim \mathbf{s}$.

Definition 2.3. An n -tabloid $\{\mathbf{t}\}$ of shape λ , also called a λ_r^n -tabloid $\{\mathbf{t}\}$, is the set of all λ_r^n -tableaux that are row-equivalent to \mathbf{t} ; i.e.,

$$\{\mathbf{t}\} = \{\mathbf{s} \mid \mathbf{s} \sim \mathbf{t}\}.$$

Let $\lambda \vdash r$ with $0 \leq r \leq n$, N^λ the F -vector space generated by all λ_r^n -tableau. We can define an action of R_n on N^λ by first determining how R_n acts on the basis of λ_r^n -tableau and then linearly extending this action to the whole space. If \mathbf{t} is a λ_r^n -tableau, for each $\pi \in R_n$ we define $\pi\mathbf{t}$ to be the zero vector if there exists $\mathbf{t}_{ij} \in \text{cont}(\mathbf{t})$ such that $\pi(\mathbf{t}_{ij}) = 0$; otherwise, we say that $(\pi\mathbf{t})_{ij} = \pi(\mathbf{t}_{ij})$. Let us rephrase this action by the language of rook n -diagram. Let $D = d_1^{-1}p_1p_2 \dots p_sd_2$ be a rook n -diagram as that in Proposition 2.1. Then $D\mathbf{t} = 0$ if there exists an integer i with $1 \leq i \leq s$ such that $(i)d_2 \in \text{cont}(\mathbf{t})$. Otherwise (in this case

$\text{cont}(\mathbf{t}) \subseteq \{(s+1)d_2, \dots, (n)d_2\}$, D maps the entry which equal to $(j)d_2$ to $(j)\sigma^{-1}d_1$ for some $s+1 \leq j \leq n$.

Let M^λ be the F -vector space generated by all λ_r^n -tabloids. If two λ_r^n -tableaux \mathbf{s}, \mathbf{t} satisfies $\mathbf{s} \sim \mathbf{t}$, then $\pi\mathbf{s} \sim \pi\mathbf{t}$ for all $\pi \in R_n$. Therefore we have an induced action of R_n on M^λ given by

$$\pi\{\mathbf{t}\} = \begin{cases} 0 & \text{if } \pi\mathbf{t} = 0, \\ \{\pi\mathbf{t}\} & \text{otherwise.} \end{cases}$$

For a λ_r^n -tableau \mathbf{t} , let $C_{\mathbf{t}} := \mathfrak{S}_{c_1} \times \mathfrak{S}_{c_2} \times \dots \times \mathfrak{S}_{c_l}$ be the column stabiliser subgroup of \mathbf{t} in symmetric group \mathfrak{S}_n , where $l = \lambda_1$. Note that $C_{\mathbf{t}}$ does not consist of all the elements in R_n that fix the columns of \mathbf{t} . For each λ_r^n -tableau \mathbf{t} we define the following element in M^λ :

$$e_{\mathbf{t}} := \sum_{\sigma \in C_{\mathbf{t}}} \text{sgn}(\sigma) \sigma\{\mathbf{t}\},$$

where $\text{sgn}(\sigma)$ stands for the sign of the permutation σ . The element $e_{\mathbf{t}}$ is called the n -polytabloid associated with \mathbf{t} .

Lemma 2.4. ([4, Proposition 3.3]) *Suppose $\pi \in R_n$, \mathbf{t} is a λ_r^n -tableau. If $\pi\mathbf{t} = 0$, then $\pi e_{\mathbf{t}} = 0$. Otherwise, $\pi e_{\mathbf{t}} = e_{\hat{\pi}\mathbf{t}}$.*

We refer the reader to [4] for the definition of $\hat{\pi}$ which will not be used in our paper. For each partition $\lambda \vdash r$ with $0 \leq r \leq n$, let us define

$$R^\lambda := F - \text{Span}\{e_{\mathbf{t}} \mid \mathbf{t} \text{ is a } \lambda_r^n\text{-tableau}\}.$$

It is clear that the subspace S^λ of R^λ generated by those n -polytabloids $e_{\mathbf{t}}$ with $\text{cont}(\mathbf{t}) = \{1, 2, \dots, r\}$ is exactly the so-called Specht module for symmetric group \mathfrak{S}_r . Therefore Grood called R^λ the *Specht module* for rook monoid algebra FR_n . We conclude the main results of [4] as follows

Theorem 2.5. *With notations as above, $\{R^\lambda \mid \lambda \vdash r, 0 \leq r \leq n\}$ forms a complete set of pairwise non-isomorphic irreducible FR_n -modules.*

3. BLOCKS OF ROOK MONOID ALGEBRA

In this section, we will decompose the rook monoid algebra FR_n as the direct sum of blocks, the indecomposable two-sided ideals, by constructing elements in FR_n analogous to the Young symmetrizers or anti-symmetrizers of symmetric groups.

Lemma 3.1. *Let ρ_1, ρ_2, ρ_3 be the three one-dimensional representations of FR_n which are defined on generators by*

$$\begin{aligned} \rho_1(s_i) &= 1, \quad \rho_1(p_j) = 0; \\ \rho_2(s_i) &= -1, \quad \rho_2(p_j) = 0; \\ \rho_3(s_i) &= 1, \quad \rho_3(p_j) = 1, \end{aligned}$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n$. For $n \geq 2$, then up to isomorphism, ρ_1, ρ_2, ρ_3 are the only three non-isomorphic one-dimensional representations of FR_n .

Proof. Using the generators and relations for FR_n , one checks easily that ρ_1, ρ_2, ρ_3 are three one-dimensional representations of FR_n .

Suppose that Fv affords a one-dimensional representation ρ of FR_n . Since $s_1^2 = 1$ and $p_1^2 = p_1$, there are only three possibilities:

Case 1. $p_1v = 0$ and $s_1v = v$. Using the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i p_i = p_{i+1} s_i$, we deduce that $s_i v = v$ and $p_j v = 0$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Hence $\rho = \rho_1$ in this case.

Case 2. $p_1v = 0$ and $s_1v = -v$. Using the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i p_i = p_{i+1} s_i$, we deduce that $s_i v = -v$ and $p_j v = 0$ for all $1 \leq i \leq n-1$ and $1 \leq j \leq n$. Hence $\rho = \rho_2$ in this case.

Case 3. $p_1v = v$. Using the relation $s_i p_i = p_{i+1} s_i$ we get that $p_j v = v$ for all $1 \leq j \leq n$. Then using the relation $p_i s_i p_i = p_i p_{i+1}$, we deduce that $s_i v = v$ for all $1 \leq i \leq n-1$. Hence $\rho = \rho_3$ in this case. This completes the proof of the lemma. \square

It is clear that the two-sided ideal $\langle p_1 p_2 \cdots p_n \rangle$ generalized by $p_1 p_2 \cdots p_n$ corresponds to the one-dimensional representation ρ_3 . Since FR_n is semisimple with base field F of characteristic 0 [11], there are two one-dimensional two-sided ideals corresponding to the two non-isomorphic one-dimensional representations ρ_1, ρ_2 . Our first task is to construct these two one-dimensional two-sided ideals in an explicit way. Let $w \in \mathfrak{S}_n$. We denote $\ell(w)$ the length function of w , i.e., the length of any reduced expression of w .

Definition 3.2. Let $D \in \text{Rd}_n$ and (d_1, d_2, r, σ) the unique quadruple with $0 \leq r \leq n$, $d_1, d_2 \in \mathcal{D}_r$, $\sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$ and such that $D = d_1^{-1} p_1 p_2 \cdots p_r \sigma d_2$. We define $\ell(D) := \ell(d_1) + \ell(\sigma) + \ell(d_2)$ and $\text{sgn}(D) := (-1)^{r+\ell(D)}$.

When $r = 0$, the definition $\ell(D)$ coincides with the length function on \mathfrak{S}_n . For any $D_1, D_2 \in \text{Rd}_n$, note that in general

$$\text{sgn}(D_1 D_2) \neq \text{sgn}(D_1) \text{sgn}(D_2).$$

For each integer $0 \leq r \leq n$, we use $\text{Rd}_n[r]$ to denote the set of rook n -diagrams which have exactly r isolated vertices in each row. Let $R_n^{(r)}$ be the two-sided ideal of FR_n generated by $p_1 p_2 \cdots p_r$. Then there is a filtration

$$FR_n = R_n^{(0)} \supset R_n^{(1)} \supset R_n^{(2)} \supset \cdots \supset R_n^{(n)} \supset 0$$

of two-sided ideals of FR_n . We have from the multiplication of rook n -diagrams that the ideal $R_n^{(r)}$ has a basis $\text{Rd}_n[r] \cup \text{Rd}_n[r+1] \cup \cdots \cup \text{Rd}_n[n]$. For $i = 1, 2$, let $X_i \in FR_n$ be an element such that the two-sided ideal $\langle X_i \rangle$ of FR_n generated by X_i corresponds to the one-dimensional representation ρ_i .

Lemma 3.3. *The elements X_1, X_2 can be taken of the following form:*

$$X_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma + \sum_{r=1}^n \sum_{D \in \text{Rd}_n[r]} C_D D,$$

$$X_2 = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \sigma + \sum_{r=1}^n \sum_{D \in \text{Rd}_n[r]} C'_D \text{sgn}(D) D,$$

where $C_D, C'_D \in F$.

Proof. It is clear that there is an algebra epimorphism

$$\theta : FR_n \twoheadrightarrow FR_n / R_n^{(1)} \cong F\mathfrak{S}_n.$$

Since the algebras FR_n and $F\mathfrak{S}_n$ are both semisimple, it follows from Lemma 3.1 that the image of the two one-dimensional two-sided ideals corresponding to

representations ρ_1, ρ_2 under θ must be the only two non-isomorphic one-dimensional two-sided ideals of $F\mathfrak{S}_n$. The lemma now follows from the well-known results about Young symmetrizers or anti-symmetrizers of symmetric groups. \square

For an arbitrary element $a \in FR_n$, we say that the rook n -diagram D is involved in a , if D appears with nonzero coefficient when writing a as a linear combination of the basis of rook n -diagrams. The following lemma is well-known for symmetric groups.

Lemma 3.4. *Let r be an integer with $0 \leq r \leq n$. There exists a unique element $w_0 \in \mathcal{D}_r$ of maximal length $r(n-r)$. If $s_{i_1}s_{i_2}\cdots s_{i_{r(n-r)}}$ is a reduced expression of w_0 , then for any integer $0 \leq j \leq r(n-r)$, there is $s_{i_1}s_{i_2}\cdots s_{i_j} \in \mathcal{D}_r$. Conversely, for any $d \in \mathcal{D}_r$, there exists a reduced expression $s_{i_1}s_{i_2}\cdots s_{i_{r(n-r)}}$ of w_0 such that $d = s_{i_1}s_{i_2}\cdots s_{i_j}$ for some $0 \leq j \leq r(n-r)$.*

Lemma 3.5. *For each integer $1 \leq r \leq n$, and any $D_1, D_2 \in \text{Rd}_n[r]$, we have*

$$C_{D_1} = C_{D_2} \quad \text{and} \quad C'_{D_1} = C'_{D_2}.$$

Proof. We only prove $C_{D_1} = C_{D_2}$ and the other assertion $C'_{D_1} = C'_{D_2}$ can be proved similarly. If $D_1 = d_1^{-1}p_1p_2\cdots p_r\sigma d_2$ and $D_2 = d_3^{-1}p_1p_2\cdots p_r\sigma d_2$ with $d_1, d_2, d_3 \in \mathcal{D}_r$ and $\sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$, we claim that

$$C_{D_1} = C_{D_2}.$$

In fact, by Lemma 3.4 it suffices to prove that

$$C_{d_1^{-1}p_1p_2\cdots p_r\sigma d_2} = C_{s_i d_1^{-1}p_1p_2\cdots p_r\sigma d_2},$$

whenever $d_1 s_i \in \mathcal{D}_r$ with $\ell(d_1 s_i) = \ell(d_1) + 1$. Let us compare the coefficients of $D_1 = d_1^{-1}p_1p_2\cdots p_r\sigma d_2$ in both sides of the equality $s_i X_1 = X_1$. Since s_i is invertible in the rook monoid R_n , we have by the concatenation rule of rook diagrams that $s_i \text{Rd}_n[f] = \text{Rd}_n[f]$ for each integer $1 \leq f \leq n$. Hence it is clear that our claim holds.

Let $*$ be the algebra anti-automorphism of FR_n which is defined on generators by $s_i^* = s_i$, $p_j^* = p_j$ for each $1 \leq i \leq n-1$ and $1 \leq j \leq n$ (see [1] for example). Using Lemma 3.3, we see that $X_1^* = X_1$ and $X_2^* = X_2$. Now combining the fact $X_1^* = X_1$ and the above claim, we have

$$C_{d_1^{-1}p_1p_2\cdots p_r\sigma d_2} = C_{p_1p_2\cdots p_r\sigma d_2} = C_{p_1p_2\cdots p_r\sigma}$$

for all $d_1, d_2 \in \mathcal{D}_r$ and $\sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$. Therefore, to prove the lemma, it suffices to show that

$$C_{p_1p_2\cdots p_r\sigma_1} = C_{p_1p_2\cdots p_r\sigma_2}$$

for all $\sigma_1, \sigma_2 \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$. Equivalently, it is enough to show that $C_{p_1p_2\cdots p_r s_i \sigma} = C_{p_1p_2\cdots p_r \sigma}$ for any $\sigma \in \mathfrak{S}_{\{r+1, r+2, \dots, n\}}$ and $r+1 \leq i \leq n$ satisfying $\ell(s_i \sigma) = \ell(\sigma) + 1$. Let us compare the coefficients of $p_1p_2\cdots p_r\sigma$ in both sides of the equality $s_i X_1 = X_1$. Since s_i is invertible in the rook monoid R_n , we have $s_i \text{Rd}_n[f] = \text{Rd}_n[f]$ for each integer $1 \leq f \leq n$ and hence $C_{p_1p_2\cdots p_r s_i \sigma} = C_{p_1p_2\cdots p_r \sigma}$. This completes the proof of the lemma. \square

The following proposition is the first main result in this section, which reveals a similarity between the elements X_1, X_2 with the symmetrizer and anti-symmetrizer in the symmetric group case.

Proposition 3.6. *The elements X_1, X_2 can be taken of the following form:*

$$X_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma + \sum_{r=1}^n (-1)^r r! \sum_{D \in \text{Rd}_n[r]} D,$$

$$X_2 = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \sigma + \sum_{D \in \text{Rd}_n[1]} \text{sgn}(D) D.$$

We divide the proof of Proposition 3.6 into two lemmas for comfortable reading.

Lemma 3.7. *The element X_1 can be taken of the following form:*

$$X_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma + \sum_{r=1}^n (-1)^r r! \sum_{D \in \text{Rd}_n[r]} D.$$

Proof. By Lemma 3.5, we can take X_1 of the following form:

$$X_1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma + \sum_{r=1}^n c_r \sum_{D \in \text{Rd}_n[r]} D,$$

where $c_r \in F$ for each $1 \leq r \leq n$. It remains to compute these c_r explicitly.

We first compute c_1 . The strategy we shall use is to compare the coefficients of p_1 in both sides of the equality

$$0 = p_1 X_1 = \sum_{\sigma \in \mathfrak{S}_n} p_1 \sigma + \sum_{r=1}^n c_r \sum_{D \in \text{Rd}_n[r]} p_1 D. \quad (3.1)$$

Note that by the concatenation rule of rook diagrams $p_1 D \in \text{Rd}_n$ for all $D \in \text{Rd}_n$ and p_1 is involved in $p_1 D$ only if $D \in \text{Rd}_n[0] \cup \text{Rd}_n[1]$. If $D \in \text{Rd}_n[0]$, then it is easy to see that p_1 is involved in $p_1 D$ ($p_1 = p_1 D$ in this case) if and only if $D = 1$, the identity element of rook monoid R_n . If $D \in \text{Rd}_n[1]$, then p_1 is involved in $p_1 D$ (also $p_1 = p_1 D$ in this case) if and only if $D = p_1$. Therefore, the coefficient of p_1 in the right-hand side of the equality (3.1) is $1 + c_1$ which implies that $c_1 = -1$.

In general, suppose that $2 \leq r \leq n$. The strategy we shall use to compute c_r is to compare the coefficients of $p_1 p_2 \cdots p_r$ in both sides of the equality (3.1). We claim that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$ (in this case, $p_1 D = p_1 p_2 \cdots p_r$ since $p_1 D \in \text{Rd}_n$) only if $D \in \text{Rd}_n[r-1] \cup \text{Rd}_n[r]$. In fact, by the concatenation rule of rook diagrams, we know that $p_1 D = p_1 p_2 \cdots p_r$ only if $D \in \text{Rd}_n[0] \cup \text{Rd}_n[1] \cup \cdots \cup \text{Rd}_n[r]$. However, if $D \in \text{Rd}_n[0] \cup \cdots \cup \text{Rd}_n[r-2]$, then $p_1 D$ have at most $r-1$ isolated vertices in each row and this proves our claim.

If $D \in \text{Rd}_n[r-1]$, then (using the concatenation rule of rook diagrams) it is clear that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$, i.e., $p_1 D = p_1 p_2 \cdots p_r$, if and only if there exists an integer $1 \leq i \leq r$ such that

- (1) $\{1, i^-\}$ is an edge of D ; and
- (2) $\{j, j^-\}$ is an edge of D for all $r+1 \leq j \leq n$; and
- (3) all other vertices of D are isolated vertices.

In this case, the number of such rook n -diagram D is r .

If $D \in \text{Rd}_n[r]$, it is easy to see that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$, i.e., $p_1 D = p_1 p_2 \cdots p_r$, if and only if $D = p_1 p_2 \cdots p_r$.

Therefore, the coefficient of $p_1 p_2 \cdots p_r$ in the right-hand side of the equality (3.1) is $r c_{r-1} + c_r$, which implies that $c_r = (-1) r c_{r-1} = (-1)^r r!$ by a simple induction argument. \square

Lemma 3.8. *The element X_2 can be taken of the following form:*

$$X_2 = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)\sigma + \sum_{D \in \text{Rd}_n[1]} \text{sgn}(D)D.$$

Proof. Our proof is similar with that given in the above lemma, but we add it here for completeness. By Lemma 3.5, we can take X_2 of the following form:

$$X_2 = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)\sigma + \sum_{r=1}^n c'_r \sum_{D \in \text{Rd}_n[r]} \text{sgn}(D)D.$$

where $c'_r \in F$ for each $1 \leq r \leq n$. It remains to compute these c'_r explicitly.

We first compute c'_1 . The strategy we shall use is to compare the coefficients of p_1 in both sides of the equality

$$0 = p_1 X_2 = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma)p_1\sigma + \sum_{r=1}^n c'_r \sum_{D \in \text{Rd}_n[r]} \text{sgn}(D)p_1D. \quad (3.2)$$

Note that by the concatenation rule of rook diagrams $p_1 D \in \text{Rd}_n$ for all $D \in \text{Rd}_n$ and p_1 is involved in $p_1 D$ only if $D \in \text{Rd}_n[0] \cup \text{Rd}_n[1]$. If $D \in \text{Rd}_n[0]$, then it is easy to see that p_1 is involved in $p_1 D$ ($p_1 = p_1 D$ in this case) if and only if $D = 1$, the identity element of rook monoid R_n . If $D \in \text{Rd}_n[1]$, then p_1 is involved in $p_1 D$ (also $p_1 = p_1 D$ in this case) if and only if $D = p_1$. Therefore, the coefficient of p_1 in the right-hand side of the equality (3.2) is $1 + (-1)c'_1$, since $\text{sgn}(p_1) = -1$, which implies that $c'_1 = 1$.

In general, suppose that $2 \leq r \leq n$. The strategy we shall use to compute c'_r is to compare the coefficients of $p_1 p_2 \cdots p_r$ in both sides of the equality (3.2). We have known that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$ only if $D \in \text{Rd}_n[r-1] \cup \text{Rd}_n[r]$.

If $D \in \text{Rd}_n[r-1]$, then (using the concatenation rule of rook diagrams) it is clear that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$, i.e., $p_1 D = p_1 p_2 \cdots p_r$, if and only if there exists an integer $1 \leq i \leq r$ such that

- (1) $\{1, i^-\}$ is an edge of D ; and
- (2) $\{j, j^-\}$ is an edge of D for all $r+1 \leq j \leq n$; and
- (3) all other vertices of D are isolated vertices.

Let D be such a rook diagram. Then $\ell(D) = (r-1) + (r-i)$ and hence $\text{sgn}(D) = (-1)^{r-i}$. At the same time, the number of such rook n -diagram is r .

If $D \in \text{Rd}_n[r]$, it is easy to see that $p_1 p_2 \cdots p_r$ is involved in $p_1 D$, i.e., $p_1 D = p_1 p_2 \cdots p_r$, if and only if $D = p_1 p_2 \cdots p_r$. In this case, $\text{sgn}(D) = (-1)^r$.

Therefore, the coefficient of $p_1 p_2 \cdots p_r$ in the right-hand side of the equality (3.2) is

$$c'_{r-1} \sum_{i=1}^r (-1)^{r-i} + c'_r (-1)^r,$$

which implies that $c'_2 = \cdots = c'_n = 0$ by a simple induction argument. \square

Proof of Proposition 3.6. This follows immediately from Lemma 3.7 and Lemma 3.8. \square

For convenience, we call X_1 , X_2 the symmetrizer and anti-symmetrizer respectively of rook monoid R_n . Now we turn to study the decomposition of the rook monoid algebra FR_n . Let S be a subset of $\{1, 2, \dots, n\}$. We define

$$\text{Rd}_S := \{D \in \text{Rd}_n \mid \{j, j^-\} \text{ is an edge of } D \text{ for } j \notin S\}.$$

It is easy to verify that the set Rd_S forms a submonoid of R_n and we denote this submonoid as R_S , which is isomorphic to rook monoid $R_{|S|}$, where $|S|$ means the coordinate of S . Especially, we consider R_i as the submonoid $R_{\{1,2,\dots,i\}}$ with $1 \leq i \leq n$. Furthermore, we write X_S (resp. Y_S) the symmetrizer (resp. anti-symmetrizer) of the rook monoid R_S . If s_{ij} is the simple transposition which interchanges i and j with $i, j \in S$, then $s_{ij}X_S = X_S s_{ij} = X_S$, $s_{ij}Y_S = Y_S s_{ij} = -Y_S$ by definition. If $i \in S$, then $p_i X_S = X_S p_i = 0$, $p_i Y_S = Y_S p_i = 0$ by definition.

Let $\lambda \vdash r$ be a partition with $0 \leq r \leq n$. For a given λ_r^n -tableau \mathbf{t} , recall that \mathcal{R}_i (resp. \mathcal{C}_j) is the set of entries in the i -th row (resp. the j -th column) of \mathbf{t} .

Definition 3.9. Let $\lambda \vdash r$ with $0 \leq r \leq n$ and \mathbf{t} a λ_r^n -tableau. Define

$$e(\mathbf{t}) := Y_{\mathcal{C}_1} Y_{\mathcal{C}_2} \cdots Y_{\mathcal{C}_{\lambda_1}} X_{\mathcal{R}_1} X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} \prod_{i \notin \text{cont}(\mathbf{t})} p_i.$$

It is helpful to point out that $e(\mathbf{t}) \neq 0$. In fact, let $\phi : FR_n \rightarrow FR_n / R_n^{(n-r+1)}$ be the canonical epimorphism. Then the image $\phi(e(\mathbf{t})) \neq 0$ by the representation theory of symmetric group (see [14, §14.7]). Note that when $r = 0$, $e(\emptyset) = p_1 p_2 \cdots p_n$, where \emptyset denotes the empty partition. For a partition $\lambda \vdash r$ with $0 \leq r \leq n$, recall that R^λ is the Specht module corresponding to λ (see the Section 2.2).

Lemma 3.10. Let $\lambda \vdash r$ be a partition with $0 \leq r \leq n$ and \mathbf{t} a λ_r^n -tableau. For any partition $\mu \vdash f$ with $0 \leq f \leq n$, we have $e(\mathbf{t})R^\mu = 0$ unless $\lambda = \mu$.

Proof. Let a be an arbitrary nonzero element of R^μ . We denote $T(\mu)$ the set of all μ_f^n -tableau for convenience. Write a as a linear combination of n -polytabloids:

$$a = \sum_{\mathbf{s} \in T(\mu)} a_{\mathbf{s}} e_{\mathbf{s}},$$

where $a_{\mathbf{s}} \in F$. We now compute $e(\mathbf{t})a$ and there are three cases occurring.

Case 1. $|\mu| > |\lambda|$. In this case, for an arbitrary μ_f^n -tableau \mathbf{s} , there exists an integer $i \in \text{cont}(\mathbf{s})$ but $i \notin \text{cont}(\mathbf{t})$. Hence $p_i \mathbf{s} = 0$ (see the paragraph below Definition 2.3). It follows from the Definition 3.9 and Lemma 2.4 that $e(\mathbf{t})e_{\mathbf{s}} = e(\mathbf{t})p_i e_{\mathbf{s}} = 0$ and hence $e(\mathbf{t})a = 0$.

Case 2. $|\mu| < |\lambda|$. From the proof of Case 1, we have

$$e(\mathbf{t})a = e(\mathbf{t}) \sum_{\mathbf{s} \in T(\mu), \text{cont}(\mathbf{s}) \subseteq \text{cont}(\mathbf{t})} a_{\mathbf{s}} e_{\mathbf{s}}.$$

Since $|\mu| < |\lambda|$, for an arbitrary μ_f^n -tableau \mathbf{s} , there exists an integer $j \in \text{cont}(\mathbf{t})$ but $j \notin \text{cont}(\mathbf{s})$. Hence $p_j \mathbf{s} = \mathbf{s}$ and thus $p_j \{\mathbf{s}\} = \{\mathbf{s}\}$ by the action of R_n on M^μ . Note that each term in the linear combination of tabloids that form $e_{\mathbf{s}}$ contains the exact same entries as \mathbf{s} . Therefore,

$$p_j e_{\mathbf{s}} = \sum_{\sigma \in C_{\mathbf{s}}} \text{sgn}(\sigma) p_j(\sigma \{\mathbf{s}\}) = \sum_{\sigma \in C_{\mathbf{s}}} \text{sgn}(\sigma) p_j \{\sigma \mathbf{s}\} = e_{\mathbf{s}}.$$

On the other hand, the elements $X_{\mathcal{R}_1}, X_{\mathcal{R}_2}, \dots, X_{\mathcal{R}_{\ell(\lambda)}}$ pairwise commute with each other. Since $j \in \text{cont}(\mathbf{t})$, there is $X_{\mathcal{R}_1} X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} p_j = 0$. Hence for an arbitrary μ_f^n -tableau \mathbf{s} , we have

$$e(\mathbf{t})e_{\mathbf{s}} = e(\mathbf{t})(p_j e_{\mathbf{s}}) = Y_{\mathcal{C}_1} Y_{\mathcal{C}_2} \cdots Y_{\mathcal{C}_{\lambda_1}} (X_{\mathcal{R}_1} X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} p_j) \prod_{i \notin \text{cont}(\mathbf{t})} p_i e_{\mathbf{s}} = 0.$$

Then $e(\mathbf{t})a = 0$ when $|\mu| < |\lambda|$.

Case 3. $|\mu| = |\lambda|$. From the proof of Case 1, we have

$$\begin{aligned} e(\mathbf{t})a &= e(\mathbf{t}) \sum_{\mathbf{s} \in T(\mu), \text{cont}(\mathbf{s}) \subseteq \text{cont}(\mathbf{t})} a_{\mathbf{s}} e_{\mathbf{s}} \\ &= e(\mathbf{t}) \sum_{\mathbf{s} \in T(\mu), \text{cont}(\mathbf{s}) = \text{cont}(\mathbf{t})} a_{\mathbf{s}} e_{\mathbf{s}}. \end{aligned}$$

Recall that $C_{\mathbf{t}}$ is the column stabiliser subgroup of \mathbf{t} in symmetric group \mathfrak{S}_n . Let $R_{\mathbf{t}} := \mathfrak{S}_{\mathcal{R}_1} \times \mathfrak{S}_{\mathcal{R}_2} \times \cdots \times \mathfrak{S}_{\mathcal{R}_{\ell(\lambda)}}$ be the row stabiliser subgroup of \mathbf{t} in symmetric group \mathfrak{S}_n . At the same time, for any $\mathbf{s} \in T(\mu)$ satisfying $\text{cont}(\mathbf{s}) = \text{cont}(\mathbf{t})$, we have $p_j e_{\mathbf{s}} = 0$ for all $j \in \text{cont}(\mathbf{t})$ by Lemma 2.4. Hence it follows from the definition of $e(\mathbf{t})$ that

$$e(\mathbf{t})a = \sum_{q \in C_{\mathbf{t}}} \sum_{p \in R_{\mathbf{t}}} \text{sgn}(q) qp \sum_{\mathbf{s} \in T(\mu), \text{cont}(\mathbf{s}) = \text{cont}(\mathbf{t})} a_{\mathbf{s}} e_{\mathbf{s}}.$$

Now by the representation theory of symmetric group (see [14, §14.7]), we have $e(\mathbf{t})a = 0$ unless $\lambda = \mu$. This completes the proof of the lemma. \square

Let $\lambda \vdash r$ be a partition with $0 \leq r \leq n$. Let \mathbf{t}^λ be the λ_r^n -tableau in which the numbers $1, 2, \dots, r$ appear in order along successive rows.

Definition 3.11. Let $\lambda \vdash r$ with $0 \leq r \leq n$. Define a two-sided ideal

$$I(\lambda) := FR_n e(\mathbf{t}^\lambda) FR_n.$$

We now at the position to give out the decomposition of rook monoid algebra FR_n as the direct sum of blocks.

Theorem 3.12. For each partition $\lambda \vdash r$ with $0 \leq r \leq n$, $I(\lambda)$ is a minimal two-sided ideal and the rook monoid algebra can be decomposed as

$$FR_n = \bigoplus_{r=0}^n \bigoplus_{\lambda \vdash r} I(\lambda).$$

Proof. Munn proved that the rook monoid algebra FR_n is semisimple in [11, Theorem 3.1]. By the well-known Wedderburn-Artin Theorem and Theorem 2.5, there is

$$FR_n = \bigoplus_{r=0}^n \bigoplus_{\lambda \vdash r} I_\lambda \cong \bigoplus_{r=0}^n \bigoplus_{\lambda \vdash r} M_{n_\lambda}(D_\lambda), \quad (3.3)$$

where I_λ is a simple subalgebra which is isomorphic to the full matrix algebra $M_{n_\lambda}(D_\lambda)$ of degree n_λ and D_λ is a finite dimensional division algebra over F . Furthermore, $M_{n_\lambda}(D_\lambda)$ corresponds to the irreducible Specht module R^λ , i.e., R^λ is the unique (up to isomorphism) irreducible module of simple algebra I_λ . On the other hand, East [1] proved that the rook monoid algebra FR_n is a cellular algebra in the sense of Graham and Lehrer [3]. By the general theory of cellular algebra [3, Proposition 2.6 and Theorem 3.4], $\text{End}_{FR_n}(R^\lambda) \cong F$ for all $\lambda \vdash r$ with $0 \leq r \leq n$, i.e., each Specht module R^λ is absolutely irreducible. That means the field F is a splitting field for FR_n and hence $D_\lambda = F$.

Let $\lambda \vdash r$ be a partition with $0 \leq r \leq n$ and $\mathbf{t}, \mathbf{s} \in T(\lambda)$. We have from the equation (3.3) that $I_\lambda R^\mu = 0$ unless $\lambda = \mu$ and $I_\lambda R^\lambda = R^\lambda$. Note that the nonzero ideal $\langle e(\mathbf{t}) \rangle$ is a sum of certain ideals I_μ . Then it follows from Lemma 3.10 that the

nonzero two-sided ideal $\langle e(\mathfrak{t}) \rangle = I_\lambda$ and hence $\langle e(\mathfrak{t}) \rangle = \langle e(\mathfrak{s}) \rangle$ is minimal. Especially, $I(\lambda) = \langle e(\mathfrak{t}^\lambda) \rangle = I_\lambda$ and the theorem is proved. \square

4. PROOF OF THEOREM 1.2

In this section we shall give the main result of this paper. That is, the proof of Theorem 1.2.

For any positive integer $k \leq n$, the natural maps $s_i \mapsto s_i, p_j \mapsto p_j$ for all $1 \leq i \leq k-1$ and $1 \leq j \leq k$ extends to an algebra embedding from FR_k into FR_n , i.e., R_k considered as the submonoid $R_{\{1,2,\dots,k\}}$. From this point of view, when $m < n$ (where $m = \dim(V)$), the anti-symmetrizer $Y_{m+1} := Y_{\{1,2,\dots,m+1\}}$ of FR_{m+1} is an element of FR_n . That is

$$Y_{m+1} = \sum_{\sigma \in \mathfrak{S}_{m+1}} \text{sgn}(\sigma)\sigma + \sum_{D \in \text{Rd}_{m+1}[1]} \text{sgn}(D)D \in FR_n.$$

By Theorem 3.12, the two-sided ideal $\text{Ann}_{FR_n}(U^{\otimes n}) = \text{Ker}(\varphi)$ is a sum of certain ideals $I(\lambda)$. We define a two-sided ideal

$$I_{m+1} = \sum_{r=0}^n \sum_{\substack{\lambda \vdash r \\ \ell(\lambda) \geq m+1}} I(\lambda).$$

We shall prove Theorem 1.2 in three parts by showing that

$$\langle Y_{m+1} \rangle \subseteq \text{Ker}(\varphi) \subseteq I_{m+1} \subseteq \langle Y_{m+1} \rangle.$$

For convenience, we set $I(m, n) := \{(i_1, \dots, i_n) \mid i_j \in \{0, 1, \dots, m\}, \forall j\}$. For any $\underline{i} = (i_1, \dots, i_n) \in I(m, n)$, we write $v_{\underline{i}} = v_{i_1} \otimes \dots \otimes v_{i_n}$ for a simple tensor. Let's start the proof by a technical lemma.

Lemma 4.1. *Let $D = d_1^{-1}p_1p_2 \dots p_r\sigma d_2$ be a rook n -diagram defined in Proposition 2.1. For any simple tensor $v_{\underline{i}} \in U^{\otimes n}$, if $Dv_{\underline{i}} \neq 0$, then $Dv_{\underline{i}} = d_1^{-1}\sigma d_2 v_{\underline{i}}$.*

Proof. By Proposition 2.1, the isolated vertices in the bottom row of D are labeled by $((1)d_2)^-, ((2)d_2)^-, \dots, ((r)d_2)^-$. Hence if $Dv_{\underline{i}} \neq 0$, there is $i_{(1)d_2} = i_{(2)d_2} = \dots = i_{(r)d_2} = 0$. In this case, we have

$$\begin{aligned} Dv_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n} &= (d_1^{-1}p_1p_2 \dots p_r\sigma d_2)v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n} \\ &= d_1^{-1}\sigma p_1p_2 \dots p_r v_{i_{(1)d_2}} \otimes \dots \otimes v_{i_{(r)d_2}} \otimes v_{i_{(r+1)d_2}} \otimes \dots \otimes v_{i_{(n)d_2}} \\ &= d_1^{-1}\sigma v_{i_{(1)d_2}} \otimes \dots \otimes v_{i_{(r)d_2}} \otimes v_{i_{(r+1)d_2}} \otimes \dots \otimes v_{i_{(n)d_2}} \\ &= d_1^{-1}\sigma d_2 v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_n}. \end{aligned}$$

This completes the proof of the lemma. \square

Lemma 4.2. *With notations as above, there is $\langle Y_{m+1} \rangle \subseteq \text{Ker}(\varphi)$.*

Proof. For any simple tensor $v_{\underline{i}} \in U^{\otimes n}$, we only need to proof $Y_{m+1}v_{\underline{i}} = 0$. By the actions of rook monoids on n -tensor spaces defined in Section 2.1, we know that Y_{m+1} only acts on the first $m+1$ components of $v_{\underline{i}}$. Hence we can assume $n = m+1$ without lose of the generality. For an arbitrary simple tensor $v_{\underline{i}} = v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_{m+1}}$, if the $(m+1)$ -tuple $(i_1, i_2, \dots, i_{m+1})$ has a repeated number, for instance, $i_j = i_k$ with $j < k$, then obviously $Y_{m+1}v_{\underline{i}} = Y_{m+1}s_{jk}v_{\underline{i}} = -Y_{m+1}v_{\underline{i}}$ and hence $Y_{m+1}v_{\underline{i}} = 0$, where s_{jk} is the transposition which interchanges j and k .

Then, we assume that i_1, i_2, \dots, i_{m+1} are different with each other. Noting that $\dim(V) = m$, we can assume $v_{\underline{i}} = v_0 \otimes v_1 \otimes \dots \otimes v_m$ without lose of the generality. Therefore, for each $D \in \text{Rd}_{m+1}[1]$, $Dv_{\underline{i}} \neq 0$ implies that the first vertex in the bottom row of D is isolated. In other words, $D = d_1^{-1}p_1\sigma$ with $d_1 \in \mathcal{D}_1$, $\sigma \in \mathfrak{S}_{\{2,3,\dots,m+1\}}$. Then we have from Lemma 4.1 that

$$\begin{aligned}
Y_{m+1}v_{\underline{i}} &= \sum_{\sigma \in \mathfrak{S}_{m+1}} \text{sgn}(\sigma)\sigma v_{\underline{i}} + \sum_{D \in \text{Rd}_{m+1}[1]} \text{sgn}(D)Dv_{\underline{i}} \\
&= \sum_{\sigma \in \mathfrak{S}_{m+1}} \text{sgn}(\sigma)\sigma v_{\underline{i}} + \sum_{\substack{D=d_1^{-1}p_1\sigma \\ d_1 \in \mathcal{D}_1, \sigma \in \mathfrak{S}_{\{2,\dots,m+1\}}}} \text{sgn}(D)Dv_{\underline{i}} \\
&= \sum_{\sigma \in \mathfrak{S}_{m+1}} \text{sgn}(\sigma)\sigma v_{\underline{i}} - \sum_{\substack{d_1^{-1}\sigma \\ d_1 \in \mathcal{D}_1, \sigma \in \mathfrak{S}_{\{2,\dots,m+1\}}}} \text{sgn}(d_1^{-1}\sigma)d_1^{-1}\sigma v_{\underline{i}} \\
&= \left(\sum_{\sigma \in \mathfrak{S}_{m+1}} \text{sgn}(\sigma)\sigma - \sum_{\substack{d_1^{-1}\sigma \\ d_1 \in \mathcal{D}_1, \sigma \in \mathfrak{S}_{\{2,\dots,m+1\}}}} \text{sgn}(d_1^{-1}\sigma)d_1^{-1}\sigma \right) v_{\underline{i}} \\
&= 0,
\end{aligned}$$

where the last identity follows from the fact that $\{d^{-1} | d \in \mathcal{D}_1\}$ is a set of left coset representatives of $\mathfrak{S}_{\{1\}} \times \mathfrak{S}_{\{2,\dots,m+1\}}$ in \mathfrak{S}_{m+1} . \square

Lemma 4.3. *With notations as above, there is $\text{Ker}(\varphi) \subseteq I_{m+1}$.*

Proof. As mentioned at the beginning of this section, $\text{Ker}(\varphi)$ is a sum of certain ideals $I(\lambda)$. We equivalently show that if $I(\lambda) \not\subseteq I_{m+1}$, then $I(\lambda) \not\subseteq \text{Ker}(\varphi)$. Taking a partition λ such that $I(\lambda) \not\subseteq I_{m+1}$, there exists an integer $0 \leq r \leq n$ such that $\lambda \vdash r$ and $\ell(\lambda) \leq m$. We shall prove $I(\lambda) \not\subseteq \text{Ker}(\varphi)$ by finding a simple tensor $v_{\underline{i}} \in U^{\otimes n}$ such that $e(\mathbf{t}^\lambda)v_{\underline{i}} \neq 0$.

Let

$$v_{\underline{i}} = \underbrace{v_1 \otimes \dots \otimes v_1}_{\lambda_1 \text{ copies}} \otimes \underbrace{v_2 \otimes \dots \otimes v_2}_{\lambda_2 \text{ copies}} \otimes \dots \otimes \underbrace{v_m \otimes \dots \otimes v_m}_{\lambda_m \text{ copies}} \otimes \underbrace{v_0 \otimes \dots \otimes v_0}_{n-r \text{ copies}}.$$

For each rook n -diagram $D \in R_n^{(n-r+1)}$, it is clear that $Dv_{\underline{i}} = 0$. Therefore,

$$\begin{aligned}
e(\mathbf{t}^\lambda)v_{\underline{i}} &= \sum_{q \in C_{\mathbf{t}^\lambda}} \sum_{p \in R_{\mathbf{t}^\lambda}} \text{sgn}(q)qp v_{\underline{i}} \\
&= |R_{\mathbf{t}^\lambda}| \sum_{q \in C_{\mathbf{t}^\lambda}} \text{sgn}(q)q v_{\underline{i}},
\end{aligned}$$

where $|R_{\mathbf{t}^\lambda}| = \lambda_1! \lambda_2! \dots \lambda_m!$ is the order of the group $R_{\mathbf{t}^\lambda}$. For each $q \in C_{\mathbf{t}^\lambda}$, if $q \neq 1$, then $qv_{\underline{i}} \neq v_{\underline{i}}$, since each element of $C_{\mathbf{t}^\lambda}$ except the identity takes at least one entry of some column of \mathbf{t}^λ to a different row. Hence the coefficient of $v_{\underline{i}}$ in $e(\mathbf{t}^\lambda)v_{\underline{i}}$ is $|R_{\mathbf{t}^\lambda}| \neq 0$, and this completes the proof of this lemma. \square

Let $\lambda \vdash r$ be a partition with $0 \leq r \leq n$. Let \mathbf{t}_λ be the λ_r^n -tableau in which the numbers $1, 2, \dots, r$ appear in order along successive columns.

Lemma 4.4. *With notations as above, there is $I_{m+1} \subseteq \langle Y_{m+1} \rangle$.*

Proof. Let $\lambda \vdash r$ with $0 \leq r \leq n$ and $\ell(\lambda) \geq m+1$. We need to prove $I(\lambda) \subseteq \langle Y_{m+1} \rangle$. It follows from the proof of Theorem 3.12 that $I(\lambda) = \langle e(\mathbf{t}_\lambda) \rangle$ and hence we only need to show $e(\mathbf{t}_\lambda) \in \langle Y_{m+1} \rangle$.

Since $\ell(\lambda) \geq m+1$, the set \mathcal{C}_1 of entries in the first column of \mathbf{t}_λ is $\{1, 2, \dots, l\}$, where $l = \ell(\lambda)$. By a direct computation, we have

$$\begin{aligned} Y_{m+1}e(\mathbf{t}_\lambda) &= Y_{m+1}Y_{\mathcal{C}_1}Y_{\mathcal{C}_2} \cdots Y_{\mathcal{C}_{\lambda_1}}X_{\mathcal{R}_1}X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} \prod_{i \notin \text{cont}(\mathbf{t})} p_i \\ &= Y_{m+1}Y_lY_{\mathcal{C}_2} \cdots Y_{\mathcal{C}_{\lambda_1}}X_{\mathcal{R}_1}X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} \prod_{i \notin \text{cont}(\mathbf{t})} p_i \\ &= (m+1)!Y_lY_{\mathcal{C}_2} \cdots Y_{\mathcal{C}_{\lambda_1}}X_{\mathcal{R}_1}X_{\mathcal{R}_2} \cdots X_{\mathcal{R}_{\ell(\lambda)}} \prod_{i \notin \text{cont}(\mathbf{t})} p_i \\ &= (m+1)!e(\mathbf{t}_\lambda). \end{aligned}$$

Therefore $e(\mathbf{t}_\lambda) \in \langle Y_{m+1} \rangle$ and this completes the proof of the lemma. \square

Proof of Theorem 1.2. It follows immediately from Lemmas 4.2, 4.3 and 4.4. \square

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